Multivariate vector sampling expansions in shift invariant subspaces[☆]

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Abstract

In this paper, we study multivariate vector sampling expansions on general finitely generated shift-invariant subspaces. Necessary and sufficient conditions for a multivariate vector sampling theorem to hold are given.

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shift-invariant subspaces, super Hilbert spaces, frames
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1. Introduction and Main Results

If H is a Hilbert space, we define $H^{(q)} = H \times H \times \cdots \times H$($q$ term). Given $f = (f_1, f_2, \cdots, f_q)^T, g = (g_1, g_2, \cdots, g_q)^T \in H^{(q)}$, the inner product f and g is defined by

$$\langle f, g \rangle_{H^{(q)}} = \sum_{p=1}^{q} \langle f_p, g_p \rangle_H.$$

Let $\varphi_j = (\varphi_{j,1}, \varphi_{j,2}, \cdots, \varphi_{j,r})^T \in L^2(\mathbb{R}^d)^{(r)}, 1 \le j \le N$ be a stable generator for the shift-invariant subspace

$$V_\varphi^2 := \left\{ \sum_{j=1}^{N} \sum_{\alpha \in \mathbb{Z}^d} a_{j,\alpha} \varphi_j(\cdot - \alpha) : \left\{ a_{j,\alpha} : 1 \le j \le N, \alpha \in \mathbb{Z}^d \right\} \in \ell^2(\mathbb{Z}^d)^{(N)} \right\}.$$

i.e., the sequence $\left\{ \varphi_j(\cdot - \alpha) : 1 \le j \le N, \alpha \in \mathbb{Z}^d \right\}$ is a Riesz basis for V_φ^2. Recall that $\left\{ \varphi_j(\cdot - \alpha) : 1 \le j \le N, \alpha \in \mathbb{Z}^d \right\}$ is a Riesz basis for V_φ^2, if there

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exist two constants $A, B > 0$ such that for any $\left\{c_{j,\alpha} : 1 \le j \le N, \alpha \in \mathbb{Z}^d\right\} \in \ell^2(\mathbb{Z}^d)^{(N)}$ one has

$$A \sum_{j=1}^{N} \sum_{\alpha \in \mathbb{Z}^d} |c_{j,\alpha}|^2 \le \left\| \sum_{j=1}^{N} \sum_{\alpha \in \mathbb{Z}^d} c_{j,\alpha} \varphi_j(\cdot - \alpha) \right\|^2_{L^2(\mathbb{R}^d)^{(r)}} \le B \sum_{j=1}^{N} \sum_{\alpha \in \mathbb{Z}^d} |c_{j,\alpha}|^2.$$

We assume throughout the paper that the vector functions in the shift-invariant subspace V^2_φ are continuous on $\mathbb{R}^d$. Equivalently(see [1] or [2]), that the generator $\varphi_j, 1 \le j \le N$ is continuous on $\mathbb{R}^d$ and

$$\sup_{x \in \mathbb{R}^d} \sum_{j=1}^{N} \sum_{p=1}^{r} \sum_{\alpha \in \mathbb{Z}^d} |\varphi_{j,p}(x - \alpha)|^2 < \infty.$$

If $V^2_{\varphi_j} (1 \le j \le N)$ is defined by

$$V^2_{\varphi_j} := \left\{ \sum_{\alpha \in \mathbb{Z}^d} a_\alpha \varphi_j(\cdot - \alpha) : \left\{a_\alpha : \alpha \in \mathbb{Z}^d\right\} \in \ell^2(\mathbb{Z}^d) \right\}.$$

Then we have

$$V^2_\varphi = \sum_{j=1}^{N} V^2_{\varphi_j}.$$

Define $\mathrm{T}_{\varphi_j} : L^2\left([(j-1)/N, j/N)^d\right) \longrightarrow V^2_{\varphi_j}$, by

$$\mathrm{T}_{\varphi_j} F = \sum_{\alpha \in \mathbb{Z}^d} c_{F,j,\alpha} \varphi_j(\cdot - \alpha),$$

where $c_{F,j,\alpha} = N^{d/2} \int_{[(j-1)/N, j/N)^d} F(x) e^{2\pi N i \alpha^T \cdot x} dx$.

For convenience, in this paper we make $\chi_p(x) = \chi_{[(p-1)/N, p/N)^d}(x)$ and $e_\alpha(x) = N^{d/2} e^{-2\pi N i \alpha^T \cdot x}$.

Lemma 1.1. $\mathrm{T}_\varphi = \sum_{j=1}^{N} \mathrm{T}_{\varphi_j}$ *is an isomorphism between* $L^2[0,1)^d$ *and* V^2_φ.

Proof. Since the sequence $\left\{\varphi_j(\cdot - \alpha) : 1 \le j \le N, \alpha \in \mathbb{Z}^d\right\}$ is a Riesz basis for V^2_φ, Therefore, for any $F \in L^2[0,1)^d$, we have

$$\begin{aligned} \|\mathrm{T}_\varphi F\|^2_{L^2(\mathbb{R}^d)^{(r)}} \le B \sum_{j=1}^{N} \sum_{\alpha \in \mathbb{Z}^d} |c_{F,j,\alpha}|^2 &= B \sum_{j=1}^{N} \left\| F \chi_{[(j-1)/N, j/N]^d} \right\|^2_{L^2[0,1)^d} \\ &= B \|F\|^2_{L^2[0,1)^d}. \end{aligned}$$

Similarly, we also have

$$\|\mathrm{T}_\varphi F\|^2_{L^2(\mathbb{R}^d)^{(r)}} \geq A\|F\|^2_{L^2[0,1)^d}.$$

$\square$

Given a nonsingular matrix M with integer entries. Let $\gamma_k + M^T\mathbb{Z}^d, 1 \leq k \leq m = |\det(M)|$ be the m distinct elements of the coset space $\mathbb{Z}^d/M^T\mathbb{Z}^d$ with $\gamma_1 = 0$. Define $Q_k = M^{-T}\gamma_k/N + M^{-T}[0,1)^d/N, 1 \leq k \leq m$, we have (see [6, P.110])

$$Q_k \cap Q_{k'} = \emptyset \text{ and } \mathrm{Vol}\left(\bigcup_{k=1}^m Q_k\right) = \frac{1}{N}.$$

Thus, for any function F integrable in $[0,1/N)^d$ and $\mathbb{Z}^d/N$-periodic, we have $\int_{[0,1/N)^d} F(x)dx = \sum_{k=1}^m \int_{Q_k} F(x)dx$.

Let $g_j \in L^2[0,1)^d, 1 \leq j \leq s$, define

$$G_p(x) := \begin{bmatrix} g_1(x)\chi_p(x) & \cdots & g_1(x)\chi_p(x + M^{-T}\gamma_m/N) \\ g_2(x)\chi_p(x) & \cdots & g_2(x)\chi_p(x + M^{-T}\gamma_m/N) \\ \vdots & \vdots & \vdots \\ g_s(x)\chi_p(x) & \cdots & g_s(x)\chi_p(x + M^{-T}\gamma_m/N) \end{bmatrix}, 1 \leq p \leq N. \quad (1.1)$$

and its related constants

$$\begin{aligned} A_G &:= \min_{1\leq p\leq N} \operatorname*{ess\,inf}_{x\in[0,1/N)^d} \lambda_{min}\left[G_p^*(x)G_p(x)\right], \\ B_G &:= \max_{1\leq p\leq N} \operatorname*{ess\,sup}_{x\in[0,1/N)^d} \lambda_{max}\left[G_p^*(x)G_p(x)\right]. \end{aligned}$$

Lemma 1.2. *Suppose that $g_j \in L^2[0,1)^d, 1 \leq j \leq s$ and $G_p(x), 1 \leq p \leq N$ is its associated matrix as in (1.1). Then*

(a) *The sequence $\left\{\overline{g_j(x)}\chi_p(x)e_\alpha(M^Tx) : 1 \leq j \leq s, 1 \leq p \leq N, \alpha \in \mathbb{Z}^d\right\}$ is a complete system for $L^2[0,1)^d$ if and only if for any $1 \leq p \leq N$ the rank of the matrix $G_p(x)$ is m a.e. in $[0,1/N)^d$.*

(b) *The sequence $\left\{\overline{g_j(x)}\chi_p(x)e_\alpha(M^Tx) : 1 \leq j \leq s, 1 \leq p \leq N, \alpha \in \mathbb{Z}^d\right\}$ is a bessel sequence for $L^2[0,1)^d$ if and only if $B_G < \infty$. In this case, the optimal Bessel bound is B_G/m.*

(c) *The sequence $\left\{\overline{g_j(x)}\chi_p(x)e_\alpha(M^Tx) : 1 \leq j \leq s, 1 \leq p \leq N, \alpha \in \mathbb{Z}^d\right\}$ is a frame for $L^2[0,1)^d$ if and only if $0 < A_G \leq B_G < \infty$. In this case, the optimal frame bounds is A_G/m and B_G/m.*

(d) *The sequence* $\left\{\overline{g_j(x)}\chi_p(x)e_\alpha(M^Tx) : 1 \le j \le s, 1 \le p \le N, \alpha \in \mathbb{Z}^d\right\}$ *is a Riesz basis for* $L^2[0,1)^d$ *if and only if it is a frame and* $s = m$.

Proof. For any $F \in L^2[0,1)^d$, we have

$$\begin{aligned}
&\left\langle F(x), \overline{g_j(x)}\chi_p(x)e_\alpha(M^Tx)\right\rangle_{L^2[0,1)^d}\\
&= \int_{[0,1)^d} F(x)g_j(x)\overline{\chi_p(x)}\,\overline{e_\alpha(M^Tx)}dx\\
&= \int_{[0,1)^d} F(x)g_j(x)\chi_p(x)e^{2\pi Ni\alpha^T\cdot M^Tx}dx\\
&= \int_{[0,1)^d} F(x)\chi_p(x)g_j(x)\chi_p(x)e^{2\pi Ni\alpha^T\cdot M^Tx}dx\\
&= \int_{[0,1/N)^d} F(x)\chi_p(x)g_j(x)\chi_p(x)e^{2\pi Ni\alpha^T\cdot M^Tx}dx\\
&= \sum_{k=1}^m \int_{Q_k} F(x)\chi_p(x)g_j(x)\chi_p(x)e^{2\pi Ni\alpha^T\cdot M^Tx}dx\\
&= \int_{M^{-T}[0,1/N)^d} \sum_{k=1}^m (F\chi_p)(x+M^{-T}\gamma_k/N) \times\\
&\qquad (g_j\chi_p)(x+M^{-T}\gamma_k/N)N^{d/2}e^{2\pi Ni\alpha^T\cdot M^Tx}dx. \qquad (1.2)
\end{aligned}$$

where we have considered the $\mathbb{Z}^d/N$-periodic extension of F. Then

$$\begin{aligned}
&\sum_{j=1}^s \sum_{\alpha\in\mathbb{Z}^d} \left|\left\langle F(x), \overline{g_j(x)}\chi_p(x)e_\alpha(M^Tx)\right\rangle_{L^2[0,1)^d}\right|^2\\
&= \frac{1}{m}\sum_{j=1}^s \Bigg\| \sum_{k=1}^m (F\chi_p)(x+M^{-T}\gamma_k/N) \times\\
&\qquad (g_j\chi_p)(x+M^{-T}\gamma_k/N)\Bigg\|^2_{L^2\left(M^{-T}[0,1/N)^d\right)}.
\end{aligned}$$

Denoting $\mathbb{F}_p(x) := \big((F\chi_p)(x), \cdots, (F\chi_p)(x+M^{-T}\gamma_m/N)\big)^T$, the above reads

$$\begin{aligned}
&\sum_{j=1}^s \sum_{\alpha\in\mathbb{Z}^d} \left|\left\langle F(x), \overline{g_j(x)}\chi_p(x)e_\alpha(M^Tx)\right\rangle_{L^2[0,1)^d}\right|^2\\
&= \frac{1}{m}\|G_p(x)\mathbb{F}_p(x)\|^2_{L^2\left(M^{-T}[0,1/N)^d\right)^{(s)}}. \qquad (1.3)
\end{aligned}$$

Since $\gamma_1, \gamma_2, \cdots, \gamma_m$ are representatives of distinct cosets of $\mathbb{Z}^d/M^T\mathbb{Z}^d$, therefore for any $1 \le k \le m$ the matrix $G(x + M^{-T}\gamma_k/N)$ has the same columns of $G(x)$. Hence, for any $1 \le p \le N$ $\operatorname{rank} G_p(x) = m$ a.e. in $[0,1/N)^d$ if and only if $\operatorname{rank} G_p(x) = m$ a.e. in $M^{-T}[0,1/N)^d$. Moreover, we have

$$\operatorname*{ess\,inf}_{x\in[0,1/N)^d} \lambda_{min}\left[G_p^*(x)G_p(x)\right] = \operatorname*{ess\,inf}_{x\in M^{-T}[0,1/N)^d} \lambda_{min}\left[G_p^*(x)G_p(x)\right]$$

and

$$\operatorname*{ess\,sup}_{x\in[0,1/N)^d} \lambda_{min}\left[G_p^*(x)G_p(x)\right] = \operatorname*{ess\,sup}_{x\in M^{-T}[0,1/N)^d} \lambda_{min}\left[G_p^*(x)G_p(x)\right]$$

To prove (a), assume that there exists set $\Omega \subseteq M^{-T}[0,1/N)^d$ with positive measure and $1 \le p_0 \le N$ such that $\operatorname{rank} G_{p_0}(x) < m$, $x \in \Omega$. Then,there exists a measurable function $v(x)$, such that $G_{p_0}(x)v(x) = 0$ and $|v(x)| = 1$ in Ω. Define $F \in L^2[0,1)^d$ such that $\mathbb{F}_{p_0}(x) = v(x)$ if $x \in \Omega$, $\mathbb{F}_{p_0}(x) = 0$ if $x \in M^{-T}[0,1/N)^d \backslash \Omega$ and $\mathbb{F}_p(x) = 0$ if $p \ne p_0$. Hence, from (1.3) we obtain the system is not complete. conversely, if the system is not complete, by using (1.3) we obtain a $\mathbb{F}_{\bar{p}}(x)$ different from 0 in a set with positive measure such that $G_{\bar{p}}(x)\mathbb{F}_{\bar{p}}(x) = 0$. Thus $\operatorname{rank} G_{\bar{p}}(x) < m$ on a set with positive measure.

If $B_G < \infty$ then, for each $F \in L^2[0,1)^d$, we have

$$\begin{aligned}
&\sum_{p=1}^{N}\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\left|\left\langle F(x), \overline{g_j(x)}\chi_p(x)e_\alpha(M^Tx)\right\rangle_{L^2[0,1)^d}\right|^2 \\
&= \sum_{p=1}^{N}\frac{1}{m}\left\|G_p(x)\mathbb{F}_p(x)\right\|^2_{L^2\left(M^{-T}[0,1/N)^d\right)^{(s)}} \\
&\le B_G\frac{1}{m}\sum_{p=1}^{N}\left\|\mathbb{F}_p(x)\right\|^2_{L^2\left(M^{-T}[0,1/N)^d\right)^{(s)}} \\
&= B_G\frac{1}{m}\sum_{p=1}^{N}\int_{M^{-T}[0,1/N)^d}\sum_{k=1}^{m}\left|(F\chi_p)(x+M^{-T}\gamma_k/N)\right|^2 dx \\
&= B_G\frac{1}{m}\sum_{p=1}^{N}\sum_{k=1}^{m}\int_{Q_k}\left|(F\chi_p)(x)\right|^2 dx \\
&= B_G\frac{1}{m}\sum_{p=1}^{N}\int_{[0,1/N)^d}\left|(F\chi_p)(x)\right|^2 dx
\end{aligned}$$

$$= \quad B_G \frac{1}{m} \sum_{p=1}^{N} \int_{[(p-1)/N,p/N)^d} |F(x)|^2 \, dx = \frac{B_G}{m} \|F\|^2_{L^2[0,1)^d}$$

Hence $\left\{ \overline{g_j(x)} \chi_p(x) e_\alpha(M^T x) : 1 \le j \le s, 1 \le p \le N, \alpha \in \mathbb{Z}^d \right\}$ is a bessel sequence for $L^2[0,1)^d$ and the optimal Bessel bound is less than or equal to $\frac{B_G}{m}$.

Let $K < B_G$. Then, there a set $\Omega_K \subset M^{-T}[0,1/N)^d$ with positive measure and $1 \le \tilde{p} \le N$ such that $\lambda_{max} \left[G^*_{\tilde{p}}(x) G_{\tilde{p}}(x) \right] \ge K$ for $x \in \Omega_K$. Let $F \in L^2[0,1)^d$ such that its associated vector function $\mathbb{F}_{\tilde{p}}(x)$ is 0 if $x \in M^{-T}[0,1/N)^d \backslash \Omega_K$, $\mathbb{F}_{\tilde{p}}(x)$ is an eigenvector of norm 1 associated with the largest eigenvalue of $G^*_{\tilde{p}}(x) G_{\tilde{p}}(x)$ if $x \in \Omega_K$ and $\mathbb{F}_p(x) = 0$ if $p \ne \tilde{p}$. Using (1.3), we obtain

$$\begin{aligned} & \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} \left| \left\langle F(x), \overline{g_j(x)} \chi_{\tilde{p}}(x) e_\alpha(M^T x) \right\rangle_{L^2[0,1)^d} \right|^2 \\ & \quad \ge \quad \frac{1}{m} \int_{M^{-T}[0,1/N)^d} K \left| \mathbb{F}_{\tilde{p}}(x) \right|^2 dx = \frac{K}{m} \|F\|^2_{L^2[0,1)^d}. \end{aligned}$$

Therefore if $B_G = \infty$ then $\left\{ \overline{g_j(x)} \chi_p(x) e_\alpha(M^T x) : 1 \le j \le s, 1 \le p \le N, \alpha \in \mathbb{Z}^d \right\}$ is not a bessel sequence for $L^2[0,1)^d$, and if $B_G < \infty$ then the optimal Bessel bound is B_G/m. This completes the proof of (b). The proofs of (c) are completely analogous.

To prove (d), we assume that $m = s$ and that the sequence is a frame. We see that it is a Riesz basis by proving that the analysis operator

$$\Lambda : L^2[0,1)^d \to \ell^2(\mathbb{Z}^d)^{(N \times s)},$$

i.e.

$$\begin{aligned} \Lambda(F) \quad := \quad & \left\{ \left\langle F(x), \overline{g_j(x)} \chi_p(x) e_\alpha(M^T x) \right\rangle_{L^2[0,1)^d} : \right. \\ & \qquad \qquad \left. 1 \le j \le s, 1 \le p \le N, \alpha \in \mathbb{Z}^d \right\}. \end{aligned}$$

is surjective (see [7, Theorem 6.5.1]). To this end, notice that when $m = s$ for any $1 \le p \le N$ the matrix $G_p, 1 \le p \le N$ is a square matrix and hence, the condition $A_G > 0$ implies that for any $1 \le p \le N$ the inverse matrix

$G_p^{-1}(x)$ exists and its entries are essentially bounded. For $1 \leq p \leq N$, let $\{c_{j,\alpha}^p : 1 \leq j \leq s, \alpha \in \mathbb{Z}^d\}$ be an element of $\ell^2(\mathbb{Z}^d)^{(s)}$. For $1 \leq p \leq N, 1 \leq j \leq s$ we define the function

$$\xi_j^p(x) := m \sum_{\alpha \in \mathbb{Z}^d} c_{j,\alpha}^p e_\alpha(M^T x),$$

and let F be the function such that

$$\mathbb{F}_p(x) = G_p^{-1}(x) \left(\xi_1^p(x), \xi_2^p(x), \cdots, \xi_s^p(x)\right)^T, x \in M^{-T}[0, 1/N)^d.$$

This function belongs to $L^2[0,1)^d$ because the entries of $G_p^{-1}(x)$ are essentially bounded. We have that $G_p(x)\mathbb{F}_p(x) = \left(\xi_1^p(x), \xi_2^p(x), \cdots, \xi_s^p(x)\right)^T$, and using (1.2) we obtain that

$$\begin{aligned} & \left\langle F(x), \overline{g_j(x)}\chi_p(x) e_\alpha(M^T x)\right\rangle_{L^2[0,1)^d} \\ = & \int_{M^{-T}[0,1/N)^d} \xi_j^p(x) N^{d/2} e^{2\pi N i \alpha^T \cdot M^T x} = c_{j,\alpha}^p. \end{aligned}$$

and consequently $\Lambda(F) = \left\{c_{j,\alpha}^p : 1 \leq j \leq s, 1 \leq p \leq N, \alpha \in \mathbb{Z}^d\right\}$.

Conversely, if $\left\{\overline{g_j(x)}\chi_p(x) e_\alpha(M^T x) : 1 \leq j \leq s, 1 \leq p \leq N, \alpha \in \mathbb{Z}^d\right\}$ is a Riesz basis. Let $\left\{f_{j,p,\alpha} : 1 \leq j \leq s, 1 \leq p \leq N, \alpha \in \mathbb{Z}^d\right\}$ be its dual Riesz basis. Then, by using(1.2) we obtain for $1 \leq p \leq N$

$$\begin{aligned} m\delta_{\alpha,0}\delta_{j,j'} = & \int_{M^{-T}[0,1/N)^d} \sum_{k=1}^m (f_{j',p,0}\chi_p)(x + M^{-T}\gamma_k/N) \times \\ & (g_j\chi_p)(x + M^{-T}\gamma_k/N) N^{d/2} e^{2\pi N i \alpha^T \cdot M^T x} dx \end{aligned}$$

Therefore, for $1 \leq j, j' \leq s$, we have

$$\sum_{k=1}^m (f_{j',p,0}\chi_p)(x + M^{-T}\gamma_k/N)(g_j\chi_p)(x + M^{-T}\gamma_k/N) = m\delta_{j,j'}, a.e.$$

Thus the matrix $G(x)$ has a right inverse; in particular, $s \leq m$. As a consequence (a) we have $s \geq m$ and, finally, $s = m$. $\square$

We consider s linear-invariant systems $\mathcal{L}_j, 1 \leq j \leq s$ in $L^2(\mathbb{R}^d)^{(r)}$ such that for any $f = (f_1, f_2, \cdots, f_r)^T \in L^2(\mathbb{R}^d)^{(r)}$,

$$(\mathcal{L}_j f)(t) = [f * P](t) = \sum_{q=1}^r \int_{\mathbb{R}^d} f_q(x) p_{j,q}(t - x) dx,$$

where $P(x)$ is an $s\times r$ matrix with entries $p_{j,q}\in L^1(\mathbb{R}^d), 1\le j\le s, 1\le q\le r$. Let $h_j(t)=\left(\overline{p_{j,1}(-t)}, \overline{p_{j,2}(-t)}, \cdots, \overline{p_{j,r}(-t)}\right)^T$, we have

$$(\mathcal{L}_j f)(t)=\langle f(\cdot), h_j(\cdot-t)\rangle_{L^2(\mathbb{R}^d)^{(r)}}.$$

The set of systems $\{\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_s\}$ is an M-stable filtering sampler for V_φ^2 if there exist two positive constants C_1 and C_2 such that [5] for any $f=f^{(1)}+f^{(2)}+\cdots+f^{(N)}\in V_\varphi^2$ where $f^{(p)}\in V_{\varphi_p}^2$, we have

$$C_1\|f\|^2_{L^2(\mathbb{R}^d)^{(r)}}\le\sum_{p=1}^{N}\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\left|\mathcal{L}_j f^{(p)}(M\alpha)\right|^2\le C_2\|f\|^2_{L^2(\mathbb{R}^d)^{(r)}}.$$

For $1\le j\le s$ and $1\le p\le N$, we define $g_{j,p}(x)$ by

$$g_{j,p}(x):=\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j\varphi_p)(\alpha)e^{-2\pi Ni\alpha^T x}. \tag{1.4}$$

Lemma 1.3. *Let f be a function in V_φ^2 such that $f=f^{(1)}+f^{(2)}+\cdots+f^{(N)}$ where $f^{(p)}\in V_{\varphi_p}^2$ and $f^{(p)}=\mathrm{T}_{\varphi_p}F_p, F_p\in L^2\left([(p-1)/N, p/N)^d\right)$. For every $1\le j\le s$, we have*

$$(\mathcal{L}_j f^{(p)})(M\beta)=\left\langle F_p(\cdot), \overline{g_{j,p}(\cdot)}e_\beta(M^T\cdot)\right\rangle_{L^2[0,1)^d},\ \beta\in\mathbb{Z}^d. \tag{1.5}$$

Proof. For each $\beta\in\mathbb{Z}^d$ we have

$$\begin{aligned}
(\mathcal{L}_j f^{(p)})(M\beta) &= \langle f^{(p)}(\cdot), h_j(\cdot-M\beta)\rangle_{L^2(\mathbb{R}^d)^{(r)}}\\
&= \left\langle\sum_{\alpha\in\mathbb{Z}^d}c_{Fp,\alpha}\varphi_p(\cdot-\alpha), h_j\left(\cdot-M\beta\right)\right\rangle_{L^2(\mathbb{R}^d)^{(r)}}\\
&= \sum_{\alpha\in\mathbb{Z}^d}c_{F_p,\alpha}\left\langle\varphi_p(\cdot-\alpha), h_j\left(\cdot-M\beta\right)\right\rangle_{L^2(\mathbb{R}^d)^{(r)}}\\
&= \sum_{\alpha\in\mathbb{Z}^d}c_{F_p,\alpha}(\mathcal{L}_j\varphi_p)(M\beta-\alpha)\\
&= \sum_{\alpha\in\mathbb{Z}^d}\langle F_p(\cdot), e_\alpha(\cdot)\rangle_{L^2\left([(p-1)/N,p/N)^d\right)}(\mathcal{L}_j\varphi_p)(M\beta-\alpha)\\
&= \left\langle F_p(\cdot), \sum_{\alpha\in\mathbb{Z}^d}\overline{(\mathcal{L}_j\varphi_p)}(M\beta-\alpha)e_\alpha(\cdot)\right\rangle_{L^2\left([(p-1)/N,p/N)^d\right)}\\
&= \left\langle F_p(\cdot), \overline{g_{j,p}(\cdot)}e_\beta(M^T\cdot)\right\rangle_{L^2\left([(p-1)/N,p/N)^d\right)}.
\end{aligned}$$

□

Theorem 1.4. *Assume that the function* $g_{j,p}(x)$ *given in (1.4) belong to* $L^\infty\left([(p-1)/N, p/N)^d\right)$ *for each* $1 \le j \le s$ *and* $1 \le p \le N$. *Let* $G_p(x)$ *be the associated matrix define in* $[(p-1)/N, p/N)^d$ *as in (1.1). The following statements are equivalents:*

(a) $A_G > 0$;

(b) *The set of systems* $\{\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_s\}$ *is an* M*-stable filtering sampler for* V_φ^2;

(c) *For* $1 \le p \le N$, *there exist vectors* $(d_1^p(x), d_2^p(x), \cdots, d_s^p(x))$ *with entries* $d_j^p \in L^\infty\left([(p-1)/N, p/N)^d\right)$ *satisfying*

$$(d_1^p(x), d_2^p(x), \cdots, d_s^p(x)) G_p(x) = (1, 0, \cdots, 0) \qquad a.e. in\, [(p-1)/N, p/N)^d; \tag{1.6}$$

(d) *There exists a frame for* V_φ^2 *having the form* $\{S_j^p(t - M\alpha) : 1 \le j \le s, 1 \le p \le N, \alpha \in \mathbb{Z}^d\}$ *such that for any* $f \in V_\varphi^2$

$$f = m \sum_{j=1}^{s} \sum_{p=1}^{N} \sum_{\alpha \in \mathbb{Z}^d} \mathcal{L}_j f^{(p)}(M\alpha) S_j^p(t - M\alpha) \quad in\, L^2(\mathbb{R}^d)^{(r)}. \tag{1.7}$$

Proof. Part (c) in Lemma 1.2 proves that conditions (a) and (b) are equivalent.

If $A_G > 0$ then for any $1 \le p \le N$, $\operatorname{ess\,inf}_{x \in [0,1/N)^d} \det\left[G_p^*(x) G_p(x)\right] > 0$ and consequently, there exists the pseudo-inverse matrix

$$G_p^\dagger(x) = \left[G_p^*(x) G_p(x)\right]^{-1} G_p^*(x).$$

Moreover, its entries are essentially bounded and its first row satisfies (1.6). Therefore, (a) implies (c).

Next, we will prove that the condition (c) implies (d). Since we have assumed that $g_{j,p}(x) \in L^\infty\left([(p-1)/N, p/N)^d\right)$ for any $1 \le j \le s$ and $1 \le p \le N$, Lemma 1.2(b) proves that

$$\left\{\overline{g_{j,p}(x)} e_\alpha(M^T x) : 1 \le j \le s, 1 \le p \le N, \alpha \in \mathbb{Z}^d\right\}$$

is a Bessel sequence in $L^2[0,1)^d$. The same argument proves that

$$\left\{m d_j^p(x) e_\alpha(M^T x) : 1 \le j \le s, 1 \le p \le N, \alpha \in \mathbb{Z}^d\right\}$$

is also a Bessel sequence in $L^2[0,1)^d$. By (1.4) and (1.6), these two Bessel sequences satisfy

$$F(x) = m \sum_{j=1}^{s} \sum_{p=1}^{N} \sum_{\alpha \in \mathbb{Z}^d} \left\langle F(\cdot), \overline{g_{j,p}(\cdot)} e_\alpha(M^T \cdot) \right\rangle d_j^p(x) e_\alpha(M^T x), F \in L^2[0,1)^d.$$

Hence, they form a pair of dual frames for $L^2[0,1)^d$(see[7, Lemma 5.6.2]). Since $S_j^p(t - M\alpha) = \mathrm{T}_\varphi[d_j^p(\cdot) e_\alpha(M^T \cdot)](t)$ and T_φ is an isomorphism, the sequence $\{S_j^p(t - M\alpha) : 1 \le j \le s, 1 \le p \le N, \alpha \in \mathbb{Z}^d\}$ is a frame for V_φ^2.

Last, we prove that the condition (d) implies (b). Notice that since we have assumed that $\left\{ \overline{g_{j,p}(x)} e_\alpha(M^T x) : 1 \le j \le s, 1 \le p \le N, \alpha \in \mathbb{Z}^d \right\}$ is a Bessel sequence with bound B_G/m and

$$(L_j f^{(p)})(M\beta) = \left\langle F(\cdot), \overline{g_{j,p}(\cdot)} e_\beta(M^T \cdot) \right\rangle_{L^2\left([(p-1)/N, p/N)^d\right)}.$$

For each $f \in V_\varphi^2$, we have

$$\sum_{p=1}^{N} \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} \left| \mathcal{L}_j f^{(p)}(M\alpha) \right|^2 \le \frac{B_G}{m} \|F\|^2_{L^2[0,1)^d} \le \frac{B_G \|\mathrm{T}_\varphi^{-1}\|_{oper}}{m} \|f\|^2_{L^2(\mathbb{R}^d)^{(r)}}.$$

If $\{S_j^p(t - M\alpha) : 1 \le j \le s, 1 \le p \le N, \alpha \in \mathbb{Z}^d\}$ is a frame for V_φ^2, then the formula (1.7) gives

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{R}^d)^{(r)}} &= m^2 \left\| \sum_{j=1}^{s} \sum_{p=1}^{N} \sum_{\alpha \in \mathbb{Z}^d} \mathcal{L}_j f^{(p)}(M\alpha) S_j^p(t - M\alpha) \right\|^2_{L^2(\mathbb{R}^d)^{(r)}} \\ &\le m^2 C \sum_{p=1}^{N} \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} \left| \mathcal{L}_j f^{(p)}(M\alpha) \right|^2, \end{aligned}$$

where C is a Bessel bound for $\{S_j^p(t - M\alpha) : 1 \le j \le s, 1 \le p \le N, \alpha \in \mathbb{Z}^d\}$. Hence, the set $\{\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_s\}$ is an M-stable filtering sampler for V_φ^2. $\square$

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